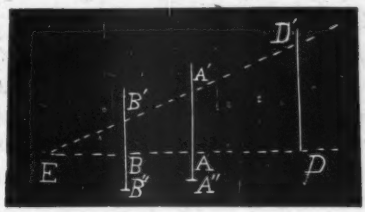
an instrument by John Robertson, which employed the runner and which was constructed in Cornhill by Messrs. Nairne and Blunt, and put upon the market. There are no indications that Robertson's rule ever became popular. Later the use of the runner was advocated by William Nicholson in an article printed in the Philosophical transactions of 1787. But in the first half of the nineteenth century I have not been able to find a single reference to the "runner" in England. It was completely forgotten.

Returning to Newton, I shall take up my third point, the early use of the slide rule in the solution of numerical equations. Oldenburg's letter to Leibniz, previously referred to, reads in translation from the Latin as follows: "Mr. Newton, with the help of logarithms graduated upon scales by placing them parallel at equal distances or with the help of concentric circles graduated in the same way, finds the roots of equations. In the arrangement of these rules all the respective coefficients lie in the same straight line. From a point of which line, as far removed from the first rule as the graduated scales are from one another, in turn, a straight line is drawn over them, so as to agree with the conditions conforming with the nature of the equation; in one of these rules is given the pure power of the required root."

If my interpretation of this passage is correct, it means in the case of the cubic $x^3+ax^2+bx=c$ that the rules A, B, D must be placed parallel and equidistant. On rule A find the number equal to the numerical value of the coefficient a; on rule B find the number equal to the numerical value of b, and on rule D find unity. Then arrange these three numbers on the rules in a straight line BD. Select the point E on this line, so that $BE=BA$. Through E pass a line ED' and turn it about E until the numbers at B', A', and D', with their proper algebraic signs attached, are seen to be together equal to the absolute term c. Then the number on the scale D' is equal to $|\ x^3\ |$, and x can be found.



Remembering that the length of $B''B$ is $\log |\ b\ |$, and assuming $BB'=\log |\ x\ |$, it follows that $B''B'$ is equal to $\log |\ bx\ |$. Then $AA'=2\log |\ x\ |$, or $\log |\ x^2\ |$ and $A''A'=\log |\ ax^2\ |$, and $DD'=\log |\ a^3\ |$. The value of x can be found by moving the scale B up until B'' reaches the point B. The number on the scale at B' will then give the numerical value of the root. A device, as represented by the line ED', fulfills some of the functions of what is now called the "runner."

In Stone's *Dictionary* (1743) Newton's scheme is modified somewhat. Stone assumes that the equation is so transformed that all its coefficients, except the absolute term, are positive, The rules are contiguous and are not all graduated alike, but have, respectively, a single, double, triple, quadruple, etc., radius. This device calls for a runner of the type now in use, carrying a thread that is at right angles to the rules. Otherwise the gen-

eral plan for the numerical solution of equations is the same as with Newton.

As a fourth point in the history of the slide rule, I desire to point out that, while so generally known to writers on the slide rule, the English astronomer William Pearson was the first one to suggest, in 1797, the inversion of the slider for certain operations with the slide rule, the inversion of fixed lines on the slide rule had been introduced more than one hundred years earlier in Everard's slide rule, used in gauging.

Finally, I desire to say a word as to the introduction of the slide rule into the United States. Brief directions for the use of the slide rule appeared in a few arithmetics imported, or reprinted in this country, in the latter part of the eighteenth century. Thus, the *Arithmetic* of George Fisher, which is a pseudonym for Mrs. Slack, probably the first woman who is the author of a popular arithmetic, contained rules for the use of the slide rule. Her books were read in the United States. In Nicolas Pike's arithmetic, an American text of 1788, such rules were given. An edition of the English book, Dilworth's *Schoolmaster's Assistant*, was brought out in Philadelphia in 1805 by Robert Patterson, professor of mathematics in the University of Pennsylvania. It devotes half a dozen pages to the use of the slide rule in gauging. Another English work, Hawney's *Complete Measurer* (1st English Edition, 1717), was printed in Baltimore in 1813. It describes the English carpenter's rule, also an English rule for gauging. Of American works, Bowditch's *Navigator*, 1802, gives one page to the explanation of the slide rule, but when working examples, Gunter's line alone is used. From these data it is difficult to draw reliable conclusions as to the extent to which the slide rule was then actually used in the United States. We surmise that it was practically unknown. The Swiss geodesist, F. R. Hassler, who came to this country and became the first superintendent of the United States Coast and Geodetic Survey, is known to have used a slide rule. The present writer had the good fortune of inspecting Hassler's slide rule. But before 1880 or 1885 it is very difficult to find references to the slide rule in American engineering literature. I have seen a reference to the slide rule in a book issued in the first half of the last century by a professor of the Rensselaer Polytechnic Institute. From this institute was graduated in 1863 Mr. Edwin Thacher, a bridge engineer, who in 1881 patented his well-known cylindrical slide rule. Interest in slide rules was awakened about this time. It was in 1881 that Robert Riddell published in Philadelphia his booklet on *The Slide Rule Simplified.* In the preface he points out that, though nearly unknown in this country, the instrument was invented before the time when William Penn founded Philadelphia. But the slide rule never became really popular in the United States until the introduction of the Mannheim type. Keuffel and Esser imported Mannheim rules in 1888 and began the manufacture of them in Jersey City in 1895. An inquiry* instituted by C. A.

**Engineering News*, Vol. 45, 1901, p. 405.

Holden in 1901 showed that in about half of the engineering schools of the United States, attention is given to the use of the slide rule.

A BIQUADRATIC EQUATION CONNECTED WITH THE REDUCTION OF A QUADRATIC LOCUS.

By DR. ARTHUR C. LUNN, The University of Chicago.

If the equation of a conic section be written in the form

$$Ax^2+By^2+2Cxy+2Dx+2Ey+F=0,$$

then it is known that a rotation of the coordinate axes through an angle a will bring them into parallelism with the axes of symmetry of the curve, provided this angle is determined by

$$\tan 2a=\frac{2C}{A-B}.$$

This rotation corresponds to the substitution

$$(1) \qquad \begin{aligned} x&=x'\cos a-y'\sin a,\\ y&=y'\cos a+x'\sin a, \end{aligned}$$

with a so chosen as to eliminate the term in $x'y'$. But the sine and cosine may be expressed in terms of the tangent of the half-angle, thus:

$$(2) \qquad t=\tan\frac{a}{2}, \quad \cos a=\frac{1-t^2}{1+t^2}, \quad \sin a=\frac{2t}{1+t^2},$$

and the use of these in (1) gives the substitution expressed rationally in terms of the parameter t. Without reference to its trigonometric source, the substitution in that form is seen to be orthogonal or rotational for all values of t, since the equation of constancy of distances:

$$(x'_1-x'_2)^2+(y'_1-y'_2)^2=(x_1-x_2)^2+(y_1-y_2)^2,$$

is directly verifiable as an identity in t.

The use of this parameter makes it possible to effect the reduction of the conic by purely algebraic processes, independently of the trigonometric formulae. For the term in $x'y'$ will have as coefficient

$$\frac{2}{(1+t^2)^2}\{2(B-A).t(1-t^2)+C[(1-t^2)^2-(2t)^2]\},$$

which will vanish if t satisfy the biquadratic equation:

$$t^4+4mt^3-6t^2-4mt+1=0, \tag{3}$$

in which is put $m=\frac{A-B}{2C}$, the trigonometric value of which is $\cot 2\alpha$. The four roots of this equation must therefore be real for all real values of m, and correspond to the four semi-axes of the conic.

But this equation must be solvable by quadratics. For, by a familiar formula of trigonometry

$$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}$$

which, regarded as a quadratic equation in $\tan\alpha$ gives

$$\tan\alpha=-\cot 2\alpha\pm\sqrt{[1+\cot^2 2\alpha]},$$

and a repetition of such solution gives $\tan\frac{1}{2}\alpha$ in terms of $\cot\alpha$. This suggests at once the four roots of the biquadratic in t, which after a little reduction prove to be:

$$\begin{aligned} t_1&=-m+r+R_1 \\ t_2&=-m-r+R_2 \\ t_3&=-m+r-R_1 \\ t_4&=-m-r-R_2 \end{aligned} \qquad \begin{aligned} &\text{where} \\ &r=\sqrt{[1+m^2]}, \\ &R_1=\sqrt{[2(r^2-rm)]}, \\ &R_2=\sqrt{[2(r^2+rm)]}. \end{aligned}$$

These roots are obviously all real, and are easily shown to be distinct. Direct computation shows that the product $(t-t_1)(t-t_2)(t-t_3)(t-t_4)$ gives the biquadratic polynomial on the left of the equation in t.

The reducibility of the biquadratic equation by quadratics is connected intimately with the existence of rational relations among the roots, which in the present case are the following:

$$t_1t_3=-1, \quad t_2t_4=-1, \quad \frac{t_2-t_1}{1+t_1t_2}=\frac{t_3-t_2}{1+t_3t_2},$$

and others (not independent) similar to the last. These correspond to the fact that, since the axes of the conic are mutually perpendicular, the various values of $\frac{1}{2}\alpha$ must be spaced at intervals of 45°.

ON A CERTAIN CLASS OF QUARTIC CURVES.*

By PROF. R. D. CARMICHAEL, Anniston, Alabama.

The object of this paper is the discussion of a class of quartic curves whose equation may be represented thus: Set

$$m_i = c_i / [(x-a_i)^2 + (y-b_i)^2], \quad i=1, 2, 3, \tag{1}$$

in which $c_i > 0$ and the radical is to be taken with the positive sign. Then put

$$(m_1+m_2+m_3)(m_1+m_2-m_3)(m_1-m_2+m_3)(m_1-m_2-m_3)=0. \tag{2}$$

It is evident that if the factors of the first member of (2) are multiplied together the resulting equation is without radicals and is of the fourth degree in x, y. Therefore its locus is a quartic curve, except in the special case in which the reduced equation breaks up into rational factors.

It should be pointed out that in the equation in its expanded form each of the c's enters to an even power, and consequently the assumption of c_i positive is no limitation on the generality of equation (2). A similar remark applies to the assumption that the radicals are positive.

Attention will be confined to a discussion of the general nature of the locus and the problem of the construction of the curve by continuous motion when the c's are commensurable.

§1. General Nature of the Locus.

Equation (2) is evidently satisfied if any one of the factors of its first member is zero. But, since c_i and the radical in (1) each is to be taken positive, it is clear that

$$m_1+m_2+m_3=0$$

can be satisfied only when $x-a_i=0$ and $y-b_i=0$; that is, when $x=a_1=a_2=a_3$, $y=b_1=b_2=b_3$. The locus of (2) is simply a single point and is therefore not properly a locus of the fourth order. We shall therefore exclude the case from further consideration.

Each of the other three cases may be represented by

$$m_\lambda = m_\mu + m_\nu, \qquad \lambda, \mu, \nu = 1, 2, 3 \text{ in some order.} \tag{3}$$

It is evident at once from (1) that the branch which is represented by (3) has this property: If P is any point on the branch, then c_λ times its distance

*Read before the American Mathematical Society, October 26, 1907.

from (a_λ, b_λ) is equal to c_μ times its distance from (a_μ, b_μ) plus c_ν times its distance from (a_ν, b_ν). Since the c's are to be taken positive and the a's, b's, and c's are to be considered finite it is evident that this condition can always be satisfied whatever the positions of the three points and the values of the c's. Hence the locus of (2) has in general three branches, and these must be distinct except for special values.

Now suppose that any two branches, say,

$$m_\lambda = m_\mu + m_\nu, \quad m_\mu = m_\lambda + m_\nu,$$

have some point in common. Then solving their equations as simultaneous, we have $m_\lambda = m_\mu$; and therefore each of the branches reduces to $m_\nu = 0$, an equation which can be satisfied only by the single point (a_ν, b_ν). That is, the two branches of the locus coincide and consist of a single point only. But, since $m_\lambda = m_\mu$, equation (2) now reduces to the form

$$(2m_\lambda + m_\nu)(2m_\lambda - m_\nu)(m_\nu)^2 = 0, \tag{4}$$

the locus of which is evidently not properly of the fourth degree. The case, therefore, in which two branches of the locus of (2) may have a common point is to be excluded from further discussion.

We propose now to find a necessary condition for the existence of an infinite branch. Suppose that

$$m_\lambda = m_\mu + m_\nu, \qquad \lambda, \mu, \nu = 1, 2, 3 \text{ in some order}, \tag{5}$$

is such a branch. Then either x or y is infinite or both are infinite for some point of the locus. Now divide equation (5) by $\sqrt{[(x-a_\lambda)^2+(y-b_\lambda)^2]}$; in the result consider the case for which some point P is infinitely removed from the origin. The limit of each of the fractions of the form

$$\frac{\sqrt{[(x-a_\mu)^2+(y-b_\mu)^2]}}{\sqrt{[(x-a_\lambda)^2+(y-b_\lambda)^2]}}$$

is easily shown to be 1 when either x or y approaches infinity or when both approach infinity. Passing to the point P and taking the limiting values of the fractions in the equation which results from the suggested division in (5), we have

$$c_\lambda = c_\mu + c_\nu, \tag{6}$$

a condition which is necessary if the locus of (5) is an infinite branch. Then, evidently, *the locus of* (2) *can in no case have more than one infinite branch,*

and for the existence of such a branch it is necessary and sufficient that a relation (6) *shall hold.*

We consider now the relative position of the closed branches of the curve. Since no two branches can have a point in common, it follows that a closed branch cannot lie partly within and partly without another branch. Suppose that one of the closed branches lies entirely within another. Then draw a line through any point within the inner of these two ovals and through some point on the third branch (considered either as an oval or as an infinite branch). Such a line cuts the quartic curve in five points; and this is impossible. Therefore, *one of the closed branches cannot lie within the other.* A similar discussion leads to the theorem that no parts of two closed branches can lie on the same straight line with any point of the third branch. The facts of this paragraph will enable one to obtain an idea of the form of the locus in each of the two possible cases which may arise.

§2. Construction by Continuous Motion.

In this section the discussion is confined to the case in which the c's are commensurable.

Since the equation of any branch may be written in the form

$$m_\lambda = m_\mu + m_\nu , \tag{7}$$

it follows that the problem of construction by continuous motion is completely solved when any branch in its most general case has been constructed.

Since the c's are now to be considered commensurable, we may multiply them by some common number d so that the resulting numbers are integers. Then let $dc_i = k_i$, k_i an integer. Consider the construction of the branch whose equation is of the form

$$dm_1 = dm_2 + dm_3 . \tag{8}$$

The coefficient of each radical in the equation is now an integer. The radicals represent the distance of a point P on the locus from A, B, and C, respectively; A, B, C being in order the points (a_1, b_1), (a_2, b_2), (a_3, b_3). Let smooth pegs be placed at A, B, and C.

Now place a pencil at P. Take a cord of convenient length and pass it around the pencil at P and the peg at A and attach it either to the pencil point or to the peg so that $dc_1 = k_1$ plies extend from P to A. (The cord is attached to the pencil if k_1 is even; to the peg, if k_1 is odd.) The unattached end passes out by the pencil at P in a convenient direction.

Take a second cord having the free end in the same direction as the free end of the first cord. Pass the other end around the pencil at P and the peg at B, leaving it yet unattached, so that $dc_2 = k_2$ plies extend from P

to B. If k_2 is even the cord passed last to the pencil; then pass it around the point C making k_3 plies between P and C, and attach the end to the pencil or to the peg at C according as k_3 is even or odd. But if k_2 is odd, the cord passed last to the peg at B; then let it pass from B to C, and then from C around the pencil at P until the requisite number, k_3, of plies extend from C to P; finally attach the end to the pencil or to the peg at C according as k_3 is odd or even.

Now let both cords be stretched tight while the pencil is held firmly at P. Then tie the free ends of the cords together at some convenient distance from the pencil so that when a pull is made on the knot both strings will be drawn tight throughout their entire lengths, with the exception of course of the free ends beyond the knot. Then if the pencil moves and the cords are kept always in the position which has been defined, it is evident that the pencil point describes the branch in consideration; for k_1 times the distance from A remains always equal to k_2 times the distance from B plus k_3 times the distance from C.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

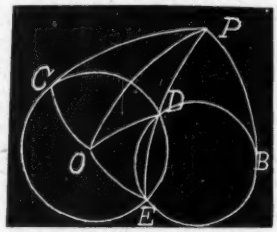
GEOMETRY.

20. Proposed by DR. GEORGE BRUCE HALSTED, Greeley, Colo.

Demonstrate by pure spherical geometry that spherical tangents from any point in the produced spherical chord common to two intersecting circles on a sphere are equal.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

No solution of this problem has yet appeared in the MONTHLY. A simple geometrical solution such as is possible for the corresponding problem in planes is not possible for this problem. The following solution is quite simple.



Let P be point on the common chord DE; PB, PC the tangents, O the pole of one circle. Let $PE=R$, $PD=r$, $PC=\rho$, $PB=\rho'$, $PO=\delta$, $OD=OE=OC=\beta$, $\angle EPO=\phi$.

Then $\cos\beta=\cos R\cos\delta+\sin R\sin\delta\cos\phi\ldots(1)$,

$\cos\beta=\cos r\cos\delta+\sin r\sin\delta\cos\phi\ldots(2)$,

$\cos\delta=\cos\beta\cos\rho\ldots(3)$,

$\cos\phi$ from (1) in (2) gives $\cos\beta(\sin R-\sin r)=\cos\delta\sin(R-r)\ldots(4)$.

$\cos\delta$ from (3) in (4) gives $\cos\rho=(\sin R-\sin r)/\sin(R-r)$.

Similarly, $\cos\rho' = (\sin R - \sin r)/\sin(R-r)$. $\therefore \rho = \rho'$.

These equations reduce to $\tan^2\frac{1}{2}\rho = \tan^2\frac{1}{2}\rho' = \tan\frac{1}{2}R \tan\frac{1}{2}r$.

Professor Philbrick gave a solution of this problem at the time it was published but it did not fill the requirements because it was not a pure spherical geometry solution.

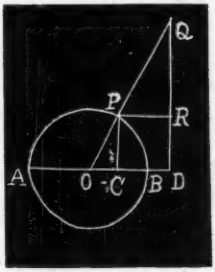
290. Proposed by J. J. QUINN, Scottdale, Pa.

(a) Suppose a circle described around the origin. Then at the end of a uniformly revolving radius r, a line equal to the diameter is pivoted. Find the equation of the locus of its extremity, if for every unit of angle its projection on the X axis is a constant linear unit, being the same part of the diameter as the angle is of π radians.

(b) Show how it can be applied to the trisection or multisection of an angle.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

(a) Let angle $POB = \theta$. Then CD, the projection of $PQ = 2r$ on AB, is $r\,\theta/90$. $OD = x = r\cos\theta + r\,\theta/90$.



$DQ = y = r\sin\theta + r\sqrt{[4-(\theta/90)^2]}$.

$\rho^2 = x^2 + y^2 = 5r^2 + 2r^2\cos\theta(\theta/90) + 2r^2\sin\theta\sqrt{[4-(\theta/90)^2]}$ is the polar equation sought.

(b) Let $m\phi$ be the angle to be multisected. $m\phi/m = \phi$. Lay off $OD = x = r\cos\phi + r\,\phi/90$. Then erect $DQ = y = r\sin\phi + r\sqrt{[4-(\phi/90)^2]}$ perpendicular to OD at D. From Q as center, with radius equal to $2r$ describe an arc cutting the circumference of the given circle at P.

Draw PO; then $\angle POD = \phi$.

300. Proposed by J. J. QUINN, Ph. D., Scottdale, Pa.

Trisect an angle by means of a tractrix.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

The length of the tangent between the axis of abscissas and the point of tangency is constant.

Let a = length of this tangent, y = an ordinate opposite angle θ, z = an ordinate opposite angle ϕ. Also let $\theta = 3\phi$.

$\therefore y = a\sin\theta = 3a\sin\phi - 4a\sin^3\phi$, $z = a\sin\phi$.

$\therefore y/z = 3 - 4\sin^2\phi$ or $\sin\phi = \frac{1}{2}\sqrt{[(3z-y)/z]}$.

$\therefore z^2/a^2 = \dfrac{3z-y}{4z}$ or $y = \dfrac{(3a^2-4z^2)z}{a^2}$.

Let $PD = z$, $PCD = \angle\phi$. Construct $QB = y = \dfrac{(3a^2-4z^2)z}{a^2}$.

Let $QAB = \theta$ where $PC = QA = a$. Then $\theta = 3\phi$.

$\therefore$ Parallel to PC draw AR, then $\angle RAB = \frac{1}{3}\angle QAB$.

$PD = z$ cannot be greater than $\frac{1}{2}a$, then $y = a$, $\theta = \frac{1}{2}\pi$, $\phi = \frac{1}{6}\pi$.

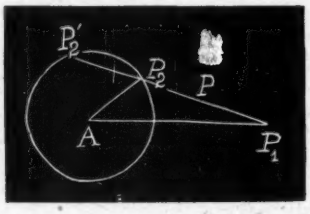
319. Proposed by S. F. NORRIS, Baltimore City College, Baltimore, Md.

Lines are drawn from a fixed point P_1, meeting a fixed circle in P_2. On P_1P_2 a point P is taken so that $P_1P \times P_1P_2 = k^2$. Find the locus of P. Solve by analytic methods, using rectangular coordinates, and putting the result in the form,

$$(x_1^2+y_1^2-r^2)[(x-x_1)^2+(y-y_1)^2]+2k^2(x_1x+y_1y-x_1^2-y_1^2)+k^4=0.$$

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let AP_1 be the axis of abscissas, A the origin of orthogonal coordinates, x and y the co-ordinates of P, ϕ the $\angle P_2P_1A$, and θ the $\angle AP_2P'_2$. Then $PP_1 = \frac{y}{\sin\phi}$, $P_1P_2 = x_1\cos\phi - r\cos\theta$. Therefore from the condition $PP_1 \times P_1P_2 = k^2$ we have $\frac{y}{\sin\phi}(x_2\cos\phi - r\cos\theta) = k^2$, but $\tan\phi = \frac{y}{x_1-x}$. $\therefore \sin\phi = \frac{y}{\sqrt{[y^2+(x_1-x)^2]}}$, $\cos\phi = \frac{x_1-x}{\sqrt{[y^2+(x_1-x)^2]}}$, $\sin\theta = \frac{x_1\sin\phi}{r}$. Substituting, we get $\frac{y}{\sin\phi}[x_1\cos\phi - \sqrt{(r^2-x_1^2\sin^2\phi)}[= k^2$, or, removing the radical, $x_1^2y^2 - 2k^2x_1\sin\phi\cos\phi\, y + k^4\sin^2\phi = r^2y^2$, and substituting now the values for $\sin\phi$ and $\cos\phi$ we obtain, after some reductions that admit of no difficulty,



$$y^2 + \left[x + \frac{k^2+r^2-x_1^2}{x_1^2-r^2}x_1\right]^2 = \frac{k^4r^2}{(x_1^2-r^2)^2},$$

or putting $x_1^2 - r^2 = a^2$ = square of the tangent drawn from P_1,

$$y^2 + \left[x + \frac{k^2-a^2}{a^2}x_1\right]^2 = \frac{k^4\, r^2}{a^4},$$

the equation of a circle, whose center lies on AP_1, at a distance $= \frac{a^2-k^2}{a^2}x_1$ from A, and whose radius $= \frac{k^2r}{a^2}$.

Also solved by G. B. M. Zerr.

CALCULUS.

247. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Integrate, $x\frac{\partial^2 y}{\partial x^2} + 2\frac{\partial y}{\partial x} - xy = 0$.

Solution by GEORGE W. HARTWELL, Columbia University, New York.

Dividing by x, $\frac{\partial^2 y}{\partial x^2}+\frac{2}{x}\frac{\partial y}{\partial x}-y=0\ldots(1)$.

Remove the term containing $\frac{\partial y}{\partial x}$ by substituting $y=\frac{v}{x}$. Then (1) becomes $\frac{\partial^2 v}{\partial x^2}-v=0\ldots(2)$.

The solution of this equation is $v=c_1e^x+c_2e^{-x}$. $\quad\therefore xy=(c_1e^x+c_2e^{-x})$.

Solved in a similar manner by G. B. M. Zerr.

248. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Evaluate $\int_0^{\frac{1}{2}\pi}\sin nx\cot x\,dx$, where n is a positive integer.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

$$A=\int_0^{\frac{1}{2}\pi}\sin nx\cot x\,dx=\int_0^{\frac{1}{2}\pi}\frac{\sin(n-1)x\,dx}{\sin x}+\int_0^{\frac{1}{2}\pi}\cos nx\,dx$$

$$=2\int_0^{\frac{1}{2}\pi}[\cos x+\cos 3x+\cos 5x+\ldots+\cos(n-3)x]\,dx+\int_0^{\frac{1}{2}\pi}\cos nx\,dx$$

$$=2\left[\sin x+\tfrac{1}{3}\sin 3x+\tfrac{1}{5}\sin 5x+\ldots+\frac{1}{n-3}\sin(n-3)x\right]_0^{\frac{1}{2}\pi}+\frac{1}{n}\cos\frac{\pi n}{2}$$

$$=2\left[1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\ldots\pm\frac{1}{n-3}\right]\pm\frac{1}{n};\ \text{for } n \text{ even}=\tfrac{1}{2}\pi \text{ if } n=\infty.$$

$$A=\int_0^{\frac{1}{2}\pi}[1+2\cos 2x+2\cos 4x+2\cos 6x+\ldots+2\cos(n-3)x]\,dx$$

$$+\int_0^{\frac{1}{2}\pi}\cos nx\,dx=\left[x+\sin 2x+\tfrac{1}{2}\sin 4x+\tfrac{1}{3}\sin 6x+\ldots+\frac{2}{n-3}\sin(n-3)x\right]_0^{\frac{1}{2}\pi}$$

$$+\frac{1}{n}\cos\frac{\pi n}{2}=\tfrac{1}{2}\pi,\ \text{for } n \text{ odd}.$$

249. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Ike, running with constant velocity v, is trying to catch Jim, running with constant velocity V, $(V>v)$, by keeping Jim dead ahead of him. Find their paths.

Solution by the PROPOSER.

Let I, J be the positions of Ike and Jim at the start; L, M their respective positions at any subsequent time t.

Let m, n be the coordinates of L; p, q the coordinates of M; ϕ the angle LM makes with IJ, $a=IJ$.

Then $dm/dt=v\cos\phi$, $dn/dt=v\sin\phi$, $p=m+a\cos\phi$, $q=n+a\sin\phi$.

$$V^2=(dp/dt)^2+(dq/dt)^2$$
$$=(dm/dt-a\sin\phi\, d\phi/dt)^2+(dn/dt+a\cos\phi\, d\phi/dt)^2$$
$$=(v\cos\phi-a\sin\phi\, d\phi/dt)^2+(v\sin\phi+a\cos\phi\, d\phi/dt)^2$$
$$=v^2+a^2(d\phi/dt)^2;\ \therefore d\phi/dt=\sqrt{(V^2-v^2)}/a=b.$$

$\therefore \phi=bt$, since $\phi=0$, when $t=0$.

$\therefore dm/dt=v\cos bt$, where $b=\sqrt{(V^2-v^2)}/a$.

$\therefore m=-\frac{v}{b}\sin bt$, $n=\frac{v}{b}(\cos bt-1)$. Therefore, Ike describes a circle.

Also, $p=a\cos bt-(v/b)\sin bt$, $q=a\sin bt+(v/b)(\cos bt-1)$.

Therefore, Jim also describes a circle.

MECHANICS.

191. Proposed by DR. L. E. DICKSON, The University of Chicago.

Give the axiomatic principle of Physics which is equivalent to the theorem on the compound of two circles ("Graphical Methods in Trigonometry," MONTHLY, June-July, 1905).

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

There are two principles that might be considered equivalent to the theorem on the compound of two circles. First, the parallelogram of velocities; second, the parallelogram of forces.

These might be named the compound of two velocities and the compound of two forces. We can state both under one theorem as follows:

The compound of a $\left\{\begin{matrix}\text{velocity}\\ \text{force}\end{matrix}\right\}$ OP with a $\left\{\begin{matrix}\text{velocity}\\ \text{force}\end{matrix}\right\}$ OR is the diagonal OQ of the parallelogram $OPQR$.

The proof by vectors follows at once. Regarding OP, OR, OQ as vectors we get at once $OP+OR=OQ$.

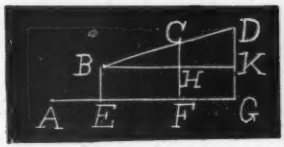
207. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

A portion of a parabola is bounded by the curve, the axis and an ordinate. A circle is inscribed to the figure which is regarded as a plane lamina. The area of the inscribed circle is now punched out. Find the centroid of what is left.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philahelphia, Pa.

Let $y^2=4ax$ be the parabola, b the abscissa of the portion considered;

r the radius of the circle. Then $(x-b+r)^2+(y-r)^2=r^2$ is the equation to the circle. Hence the invariants are $\triangle = -4a^2$, $\triangle'=-r^2$, $\theta=-4a(b-r+a)$, $\theta'=-4a(b-r)$. In order that the circle and parabola may touch $(\theta\theta'-9\triangle\triangle')^2=4(\theta^2-3\triangle\theta')\times(\theta'^2-3\triangle'\theta)$.



$\therefore 16r^5+(13a-48b)r^4+8(a^2+6b^2-14ab)r^3+8(15ab^2-2b^3-9a^2b)r^2+32ab(3ab-a^2-2b^2)r+16ab^2(b^2+a^2-2ab)=0$. This gives the value of r.

Area of semi-parabola $=\frac{4}{3}b\sqrt{(ab)}$, area of circle $=\pi r^2$, area of portion left $=\frac{4}{3}b\sqrt{(ab)}-\pi r^2$.

Let A be the vertex of the parabola, B the centroid required, C the centroid of the semi-parabola, D the centroid of the circle. Let (α, β) be the coordinates of B. The coordinates of C are $(\frac{3}{5}b, \frac{3}{4}\sqrt{ab})$; of D, $(b-r, r)$.

$$\frac{CD}{BC}=\frac{HK}{BH}=\frac{GF}{EF}=\frac{\frac{4}{3}b\sqrt{(ab)}-\pi r^2}{\pi r^2} \text{ or } \frac{EG}{EF}=\frac{4b\sqrt{(ab)}}{3\pi r^2}.$$

$$\therefore \frac{b-r-a}{\frac{3}{5}b-a}=\frac{4b\sqrt{(ab)}}{3\pi r^2}. \quad \therefore a=\frac{3[4b^2\sqrt{(ab)}+5\pi r^3-5\pi br^2]}{5[4b\sqrt{(ab)}-3\pi r^2]}.$$

$$\text{And } \frac{DK}{CH}=\frac{EG}{EF}=\frac{4b\sqrt{(ab)}}{3\pi r^2}=\frac{r-\beta}{\frac{3}{4}\sqrt{(ab)}-\beta}; \quad \therefore \beta=\frac{3(ab^2-\pi r^3)}{4b\sqrt{(ab)}-3\pi r^2}.$$

208. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, Eng.

Hanging at rest over a smooth pulley are two equal scale pans of the same mass. Two equal particles, the one inelastic and the other elastic, are simultaneously dropped from the same height one into each scale pan. Show that each impact after the first must occur when the pans have returned to the *status quo ante*, and find the total space described by either pan before motion ceases.

Remark by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

This is the same as problem 121, Mechanics. A solution of this problem is found in Vol. VIII, No. 10, pp. 203-4, October, 1901.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

141. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Given that the highest factor of a prime p contained in $m!$ is p^{m-s}; find general expressions involving p and m and s, from which, when a solution is possible, m can be determined when s is a given integer and p is a given prime. Is it then possible in any case to have more solutions than one?

Solution by the PROPOSER.

The result takes different forms according as p is 2 or is an odd prime.

First suppose $p=2$. According to Legendre (Théorie des nombres, 3rd ed., I., p. 10) the highest power of 2 contained in

$$m=2^{\alpha}+2^{\beta}+2^{\gamma}+\ldots \text{ to } s \text{ terms, } \alpha, \beta, \gamma, \ldots \text{ different integers, is } 2^{m-s};$$

therefore when $p=2$, there is an infinite number of solutions. Every m which may be expressed as a sum of s different powers of 2 will satisfy the imposed conditions.

Henceforth take p an odd prime. Suppose m expressed in the form

$$(1) \qquad m=c_n p^n+c_{n-1} p^{n-1}+\ldots+c_1 p+c_0,$$

where each c is positive and less than p, or is zero. Then in any case m is determined when the values of the c's are found.

According to Legendre (l. c.) the highest power of p contained in m is that of which the index is $\frac{m-(c_n+c_{n-1}+\ldots+c_1+c_0)}{p-1}$. But, this is the $m-s$ of the problem. Therefore,

$$m-s=\frac{m-(c_n+c_{n-1}+\ldots+c_1+c_0)}{p-1}.$$

This equation reduces to

$$m+\frac{c_n+c_{n-1}+\ldots+c_1+c_0}{p-2}=\left(1+\frac{1}{p-2}\right)s;$$

or, replacing m by its value from (1), we have

$$(2) \qquad c_n p^n+c_{n-1} p^{n-1}+\ldots+c_1 p+c_0+\frac{c_n+c_{n-1}+\ldots+c_1+c_0}{p-2}=\left(1+\frac{1}{p-2}\right)s.$$

Now, the second member of this equation is a known quantity. If it is written in the form

$$\left(1+\frac{1}{p-2}\right)s=r_v p^v+r_{v-1}p^{v-1}+\ldots+r_1 p+r_0+\frac{u}{p-2},$$

where each r is positive and less than p or is zero, and $u<p-2$, we have, from (2),

$$(3)\qquad c_n p^n+c_{n-1}p^{n-1}+\ldots+c_1p+c_0+\frac{c_n+c_{n-1}+\ldots+c_1+c_0}{p-2}$$
$$=r_v p^v+r_{v-1}p^{v-1}+\ldots+r_1p+r_0+\frac{n}{p-2},$$

from which the values of the c's (and hence by (1) the value or values of m) may be found.

As an example, take the following case: $p=5$ and $s=3725$. We have from (3),

$$c_n p^n+c_{n-1}p^{n-1}+\ldots+c_1p+c_0+\frac{c_n+c_{n-1}+\ldots+c_1+c_0}{p-2}$$
$$=5^5+2.5^4+4.5^3+3.5^2+3.5+1+\tfrac{2}{3}.$$

It is easy to show that $n=5$ and $c_5=1$; then that $c_4=2$, and $c_3=4$, and $c_2=3$, and $c_1=2$, and $c_0=2$. (Notice that the values of the c's are most easily determined in the order given.) This gives

$$m=5^5+2.5^4+4.5^3+3.5^2+2.5+2=4962.$$

It is now evident that the solution or solutions may *always* be readily obtained from (3).

When $s=5$ and $p=3$, $m=7$ or 9; and this is sufficient to show that more than one solution exists in some cases even when p is odd.

An incomplete solution was received from Professor Zerr.

143. Proposed by JOHN D. WILLIAMS (being the first of 14 challenge problems published in 1832).

Make $x^2+y^2=a^2=z^2+w^2$ and $x^2-w^2=z^2-y^2=\square$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let $x=2q(7q-5p)$, $y=25p^2-70pq+48q^2$,
$z=24p^2-70pq+50q^2$, $w=p(10q-7p)$.
$\therefore x^2+y^2=z^2+w^2=(25p^2-70pq+50q^2)^2=a^2$.
$x^2-w^2=z^2-y^2=196q^4-280pq^3+140p^3q-49p^4=(14q^2-10pq-\frac{25}{7}p^2)^2$

when $q=\frac{1513p}{1680}$; $m=1\frac{513}{1680}$, where $pm=q$.

$$\therefore x=\frac{473569\,p^2}{201600},\quad y=\frac{52319p^2}{58800},\quad z=\frac{85345p^2}{56448},\quad w=\frac{337p^2}{168}.$$

Reducing to a common denominator and expunging common factors, we get

$x=3314983p^2$, $y=1255656p^2$, $z=2133625p^2$, $w=2830800p^2$.
$\therefore x^2+y^2=z^2+w^2=a^2=(3544825p^2)^2$,
$x^2-w^2=z^2-y^2=b^2=(1725017p^2)^2$.
Also $a-x=2(339p)^2=2c^2$, $a-z=2(840p)^2=2d^2$,
$a-y=(1513p)^2=h^2$, $a-w=(845p)^2=k^2$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

General expressions for the sides of duplicate right-triangles having the same hypotenuse are

$x=(p^2-q^2)(r^2-s^2)+4pqrs,\ y=2rs(p^2-q^2)-2pq(r^2-s^2),$

$z=(r^2-s^2)(p^2+q^2),\ w=2rs(p^2+q^2).$

$\therefore x^2+y^2=z^2+w^2=[(r^2+s^2)(p^2+q^2)]^2,$ Let $p=2,\ q=1.$

$\therefore x=3(r^2-s^2)+8rs,\ y=6rs-4(r^2-s^2),\ z=5(r^2-s^2),\ w=10rs.$

$x^2+y^2=z^2+w^2=[5(r^2+s^2)]^2,\ x^2-w^2=z^2-y^2=9r^4+48r^3s-54r^2s^2-48rs^3+9s^4=\square.$ This is a square when $r=\frac{17}{12}s.$

$\therefore x=\frac{689s^2}{64},\ y=\frac{161s^2}{36},\ z=\frac{725s^2}{144},\ w=\frac{85s^2}{6}.$

$\therefore x=2067s^2,\ y=644s^2,\ z=725s^2,\ w=2040s^2.$

$x^2+y^2=z^2+w^2=(2165s^2)^2=a^2,\ x^2-w^2=z^2-y^2=(333s^2)^2=b^2.$

A solution of this problem is given in J. D. Williams' *Algebra*, page 419. He starts with $a^2=b^2+f^2=c^2+e^2$, and $b^2-c^2=d^2=e^2-f^2$. Then he is to make a^2-b^2 a square, a^2-c^2 a square, and b^2-f^2 a square. He assumes $a^2=(p^2+q^2)(r^2+s^2),\ b=pr\pm qs,\ c=ps\pm qr$. Then he assumes $r=pm-qn,\ s=pn+qm$. He finally arrives at the conclusion that $a=697,\ b=680,\ f=153,\ c=672,\ e=185$, a set of erroneous values, as Dr. Zerr has pointed out. It is likely that Williams' solution may be carried out so that a set of correct values may be obtained. Williams proposed this problem in 1832 as a challenge problem to the mathematicians of the United States. ED. F.

144. Proposed by JOHN D. WILLIAMS (being the ninth of his 14 challenge problems proposed in 1832).

Make $(m^2+n^2)^2x^2\pm(m^2+n^2)x=\square,\ (m^2-n^2)^2x^2\pm(m^2-n^2)x=\square,$ and $4m^2n^2x^2\pm2mnx=\square.$

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let $m^2+n^2=p,\ m^2-n^2=q,\ 2mn=r.$

$\therefore p^2x^2\pm px=\square,\ q^2x^2\pm qx=\square,\ r^2x^2\pm rx=\square\ldots(1,\ 2,\ 3).$

Let $p^2x^2+px=a^2x^2;\ \therefore x=p/(a^2-p^2).$

This value of x in (2) and (3) gives

$$q^2p^2+qp(a^2-p^2)=\square,\quad r^2p^2+rp(a^2-p^2)=\square\ldots(4,\ 5).$$

Let $q^2p^2+qp(a^2-p^2)=[pq-b(a-p)]^2.$

$\therefore (a-p)=\frac{2pq(b+p)}{b^2-pq}.$ This value of a in (5) gives

$$r^2b^4+4b^3pqr+2b^2pqr(2p+2q-r)+4bp^2q^2r+p^2q^2r^2=\square$$
$$=(rb+2bpq-pqr)^2,\ \text{suppose.}$$

$$\therefore b=\frac{2pqr}{pr+qr-pq};\ x=\frac{p}{a^2-p^2}=\frac{(b^2-pq)^2}{4bpq\ (b+p)\ (b+q)}.$$

$$\therefore x=\frac{-[(pr+qr-pq)^2-4pqr^2]^2}{8pqr(pr+qr-pq)(pq-pr+qr)(pq+pr-qr)}.$$

$$\therefore x=\frac{-[(4m^3n-m^4+n^4)^2-16m^2n^2(m^4-n^4)]^2}{16mn(m^4-n^4)(4m^3n-m^4+n^4)(m^4-n^4-4mn^3)(m^4-n^4+4mn^3)}$$

$$=\pm\frac{A^2}{16mn(m^4-n^4)B^2}, \text{ suppose.}$$

x is $\pm$ according as px, qx, rx is $\mp$.

$$\therefore (m^2+n^2)x^2\pm(m^2+n^2)x$$

$$=\left[\frac{A}{16mn(m^2-n^2)B}\right]^2[16m^2n^4(n^2-2m^2)+(8mn^3+n^4-m^4)(m^4-n^4)]^2. (6).$$

$$(m^2-n^2)x^2\pm(m^2-n^2)x$$

$$=\left[\frac{A}{16mn(m^2+n^2)B}\right]^2[16m^2n^4(n^2+2m^2)-(8mn^3+m^4-n^4)(m^4-n^4)]^2. (7).$$

$$4m^2n^2x^2\pm 2mnx$$

$$=\left[\frac{A}{8(m^4-n^4)B}\right][3(m^4-n^4)^2-8m^3n(m^4-n^4)-16m^2n^6]^2. (8).$$

m and n can have any values that make B positive. Let $m=2$, $n=1$; $A=671$, $B=2737$.

(6) gives $(m^2+n^2)^2x^2\pm(m^2+n^2)x=(290543/262752)^2$.

(7) gives $(m^2-n^2)^2x^2\pm(m^2-n^2)x=(74481/437920)^2$.

(8) gives $4m^2n^2x^2)\pm 2mnx=(234179/328440)^2$.

AVERAGE AND PROBABILITY.

191. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Two random lines cut a given circle. What is the chance that they intersect within the circle?

II. Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let AB, CD be the random lines, $\angle AOH=\theta$, $\angle COE=\phi$, $\angle EOH=\psi$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and θ; of ψ, $\theta-\phi$ and $\theta+\phi$ for favorable cases, and 0 and π for total cases.

Hence the chance is

$$p=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\theta-\phi}^{\theta+\phi} d\theta\, d\phi\, d\psi}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{\pi} d\theta\, d\phi\, d\psi}$$

$$=\frac{8}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\theta-\phi}^{\theta+\phi} d\theta\, d\phi\, d\psi=\frac{16}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\phi\, d\theta\, d\phi=\frac{8}{\pi^3}\int_0^{\frac{1}{2}\pi}\theta^2 d\theta=\frac{1}{3}.$$

One of our readers has again called up the question of the correctness of the two solutions, saying that he is unable to decide which is correct. We shall continue to repeat our answer as long as this question is asked. There is no such thing as *the correct solution* of the two published solutions—one is just as correct as the other. Both are correct when the law of distribution of the events assumed are granted. To these two solutions might be added an indefinite number of other solutions of equal merit. The fact of the matter is, that the problem is stated in the *indefinite* form, and when thus stated admits of an indefinite number of solutions. There is nothing to prevent one from assuming that the lines are drawn through each of two points taken at random within the circle, making it the equivalent of problem 5450 of the *Educational Times*, as was referred to by our reader. Were one to add to the problem referred to in the *Educational Times* the law of distribution of the points, then this problem would be definite, and there could be only one correct solution possible. ED. F.

MISCELLANEOUS.

169. Proposed by E. D. ROE, Ph. D., Syracuse University, Syracuse, N. Y.

Find the value for all finite values of k of

$$\lim_{x=\infty}\left[x^k \log\left(\frac{e^x+1}{e^x-1}\right)\right].$$

Solution by the PROPOSER.

1. If x is positive, it will be useful to write,

$$x^k \log\left[\frac{e^x+1}{e^x-1}\right]=e^{-x}e^x x^k \log\left[\frac{1+e^{-x}}{1-e^{-x}}\right]=\frac{x^k}{e^x}\log\left[\left(1+\frac{1}{e^x}\right)^{e^x}\left(1-\frac{1}{e^x}\right)^{-e^x}\right].$$

Therefore, $\underset{x=\infty}{L}\, x^k \log\left[\frac{e^x+1}{e^x-1}\right]=\log e^2 . \underset{x=\infty}{L}\frac{x^k}{e^x}=2L\frac{x^k}{e^x}$.

Since $\frac{x^k}{e^x}=\frac{x^k}{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots.}$, $\underset{x=\infty}{L}\frac{x^k}{e^x}=0$, for all finite values of k, and

therefore $\underset{x=\infty}{L}\, x^k \log\left[\frac{e^x+1}{e^x-1}\right]=0$.

2. If x is negative, $\underset{x=\infty}{L}\, x^k \log\left[\frac{e^x+1}{e^x-1}\right]=\underset{x=\infty}{L}\, x^k . \log(-1)=\infty$.

Also solved by G. B. M. Zerr, J. Scheffer, and S. A. Corey.

PROBLEMS FOR SOLUTION.

ALGEBRA.

295. Proposed by CHARLES GILPIN, JR., Philadelphia, Pa.

In the equation $x^3-ax\pm b=0$, we have the following relation between the coefficients and the roots: (1) When $a^3/b^2=6.75$ there are three real roots, two of which are equal; (2) when $a^3/b^2<6.75$ there are two imaginary roots and one real one; and (3) when $a^3/b^2>6.75$ there are three real, unequal roots.

296. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philahelphia, Pa.

Sum the series, $1+\frac{1}{6}+\frac{1}{20}+\frac{1}{50}+\frac{1}{105}+\frac{1}{196}+\frac{1}{336}+\ldots$

GEOMETRY.

329. Proposed by JOHN JAMES QUINN, Ph. D., Scottdale, Pa.

1. Determine the equation of the locus of a fixed point in a circle of radius r, rolling along the axis of an upright cylinder of the same radius, while the axis revolves (carrying the circle with it) through an angle equal to the central angle of the rolling circle formed by the radii to the fixed point and the point of contact.

2. Suppose the point projected into the surface of the cylinder.

3. What is the surface generated by the radius of the rolling circle?

4. What is the surface generated by a radius of the cylinder through the moving point?

CALCULUS.

252. Proposed by J. H. MEYER, S. J., Augusta, Ga.

Supposing the arc of a semi-circle to be stretched out into a straight line, and an indefinite number of perpendiculars erected on it, each equal to the versed sine of the corresponding arc; what would be the length of the curve traced out by the tops of the perpendiculars?

253. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find the maximum number of real points of inflection in each of the quartic curves $y^2=\alpha x^4\pm x^2+\beta$, and find the necessary and sufficient relations between α and β for the existence of this number of points of inflection.

MECHANICS.

212. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, Eng.

A peg A is vertically d feet above a peg B. A string AD, a feet long, with two equal, jointed rods DC, CB form the whole figure. Discuss the position of equilibrium.

213. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

Two unequal, uniform, smoothly hinged rods are placed over a smooth vertical circle. Apply the principle of vertical work to find the condition of equilibrium in terms of the length of each rod, the diameter of the circle and the angle of either rod with the vertical.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

151. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

In the recurring series, $n=0, 1, 2, 3, 4, 5, 6, 7, \ldots$
$u_n=3, 0, 2, 3, 2, 5, 5, 7, \ldots$
where the scale of relation is $u_{n+3}=u_{n+1}+u_n$, prove that u_p is always divisible by p when p is prime. Is the converse true?

AVERAGE AND PROBABILITY.

193. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

What is the average area of all squares that may be inscribed in a given sector of a circle, a diagonal of the square being parallel to a random line across the sector?

194. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

What is the mean value of the triangle formed by joining three points taken at random on the circumference of a circle?

MISCELLANEOUS.

176. Proposed by WM. E. HEAL, Coffeyville, Kansas.

In Grassman's *Extensive Algebra*, $e_1e_2=-e_2e_1$. If $e_1=e_2$, $e_1^2=-e_1^2=0$. In quaternions, $ij=-ji$, $i^2j=i.ij=ik=-j$, $i^2=-1$. Reconcile these apparently divergent results.

NOTES AND NEWS.

The next Summer meeting of the American Mathematical Society will be held at the University of Illinois during September.

On December 17th occurred the death of Lord Kelvin, one of the greatest mathematicians and physicists of the present age.

On the 7th of November, 1907, occurred the death of Professor J. R. Rand, Professor of Mathematics in Bates College, Lewiston, Me. His successor is George E. Ramsdell.

During the last convocation week, Professor G. A. Miller was elected vice president of the American Mathematical Society, chairman of the Chicago Section of this Society, and secretary of Section A of the American Association for the Advancement of Science. The last position is for five years, while the other two are for one year only.

BOOKS.

Irrational Numbers and their Representation by Sequences and Series. By Henry Parker Manning, Ph. D., Assistant Professor of Pure Mathematics in Brown University. 12mo. Cloth, vi+123 pages. Price, $1.25. New York: John Wiley & Sons.

In this book, the author has given an explanation of irrational numbers and those parts of algebra which depend on the Theory of Limits. An *irrational number* is defined by the position it occupies among rational numbers, the definition given by Dedekind.

The book contains five chapters, of which the first deals with Irrational Numbers; the second, Sequences; the third, Series; the fourth, Power Series; and the fifth, the Exponential, Binomial, and Logarithmic Series.

The work will prove very helpful to all teachers of mathematics who wish to know something of the modern views of irrational numbers and series. F.

Introduction to Higher Algebra. By Maxime Bôcher, Professor of Mathematics in Harvard University. Prepared for publication with the cooperation of E. P. R. Duval, Instructor in Mathematics in the University of Wisconsin. 8vo. Cloth, xi+321 pages. Price, $2.00. New York: The Macmillan Co.

The object of this book is to "introduce the student to higher algebra in such a way that he shall, on the one hand, learn what is meant by a proof in algebra and acquaint himself with the proofs of the most fundamental facts, and, on the other, become familiar with many important results of algebra which are new to him."

The book contains twenty-two chapters. Some notion of what the work contains may be gained from the subject treated under each of the chapters. Chapter I treats of Polynomials and their most Fundamental Properties; Chapter II, a Few Properties of Determinants; Chapter III, The Theory of Linear Dependence; Chapter IV, Linear Equations; Chapter V, Some Theorems concerning the Rank of a Matrix; Chapter VI, Linear Transformations and the Combination of Matrices; Chapter VII, Invariants,—First Principles and Forms; Chapter VIII, Bilinear Forms; Chapter IX, Geometric Introduction to the Study of Quadratic Forms; Chapter X, Quadratic Forms; Chapter XI, Real Quadratic Forms; Chapter XII, The System of a Quadratic Form and One or More Linear Forms; Chapter XIII, Pairs of Quadratic Forms; Chapter XIV, Some Properties of Polynomials in General; Chapter XV, Factors and Common Factors of Polynomials in One Variable and of Binary Forms; Chapter XVI, Factors of Polynomials in Two or More Variables; Chapter XVII, General Theorems on Integral Rational Invariants; Chapter XVIII, Symmetric Polynomials; Chapter XIX, Polynomials Symmetric in Pairs of Variables; Chapter XX, Elementary Divisors and the Equivalence of λ-Matrices; Chapter XXI, The Equivalence and Classification of Pairs of Bilinear Forms and Collineations; Chapter XXII, The Equivalence and Classification of Pairs of Quadratic Forms.

While, in the reading of this book, no algebraic knowledge is presupposed beyond a

familiarity with elementary algebra up to and including quadratics, yet familiarity with some of the higher branches of mathematics, as Analytical Geometry and the Calculus, is essential to understand and master it. The work, as its name implies, is a splendid introduction not only to algebra alone, but to a general course in mathematics. F.

Plane and Spherical Trigonometry. By A. H. Buchanan, LL. D., Professor of Mathematics, Cumberland University. 8vo. Cloth, v+96 pages, 33 figures. Price, $1.00. New York: John Wiley & Sons.

This book aims to comprise about all of trigonometry that is required in most colleges for the A. B. degree. Some of the demonstrations of principles differ slightly from those usually given. The tables are bound in a separate volume. The book, from the printer's stand point, is well gotten up. F.

A Course in Mathematics For Students in Engineering and Applied Science. By Frederick S. Woods and Frederick H. Bailey, Professors of Mathematics in the Massachusetts Institute of Technology. Vol. I. Algebraic Equations, Functions of One Variable, Analytic Geometry, Differential Calculus. 8vo. Cloth, xii+385 pages. Price, $2.00. Boston and Chicago: Ginn & Co.

This book, an entirely new venture, is the first volume of a course in mathematics designed to present in a somewhat more closely connected manner, an amount of mathematical material usually given in distinct courses under the subjects of Algebra, Analytical Geometry, Differential Calculus, and Differential Equations, and covers the work usually required of students in the first two years in colleges and engineering schools. It is believed that such a presentation of the subject will give the student a better grasp of mathematics as a whole. The results of this method of presenting mathematics will be watched with great interest by teachers of mathematics. The book is well gotten up and the material is well arranged and well selected. F.

The Metric and British Systems of Weights, Measures, and Coinage. By F. Mollwo Perkin, Ph. D., Head of Chemistry Department, Borough Polytechnic Institute, London. With Seventeen Diagrams. 8vo. Cloth, 83 pages. Price, 50 cents. London: Whittaker & Co. New York: The Macmillan Co.

This little book will be found of great value in the hands of students studying Chemistry, Physics, Engineering, or General Elementary Science.

In the introduction, the author makes a plea for the use and introduction of the metric system in the British Empire. In several tables of length, area, solid contents, etc., conversion tables are given. Also a chapter on specific gravities is inserted. F.

Integration by Trigonometric and Imaginary Substitution. By Charles O. Gunther, M. E., Assistant Professor of Mathematics and Mechanics in Stevens Institute of Technology, with an Introduction by J. Burkett Webb, C. E., Professor of Mathematics and Mechanics in Stevens Institute of Technology. 8vo. Cloth, vi+79 pages. Price, $1.25. New York: D. Van Nostrand Co.

In this little book the author, by means of what he calls the "Triangle Method," eliminates the *Reduction Formulae.* The student, by the use of this method, becoming practically independent of reduction formulae and tables of integrals. The method is worthy the attention of writers on the Integral Calculus. F.

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THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XV. FEBRUARY, 1908. NO. 2.

EXTERNAL ENCOURAGEMENT FOR THE STUDY OF HIGHER MATHEMATICS.

By DR. G. A. MILLER, University of Illinois.

Among the potent external encouragements for the study of higher mathematics are the international congresses, national societies, prizes, fellowships, and scholarships. In our own country the last two relate mainly to those who are beginners in original mathematical work while the others affect principally those who are continuing such work after the so-called student days. The first formal international mathematical congress was held in Zürich, Switzerland, in 1897, and it was then resolved to hold such congresses about every three years in various countries. Thus far only two others have been held,—one in Paris, France, in 1900, and the other in Heidelberg, Germany, in 1904. The fourth is to be held in Rome, Italy, in April of the present year. The interest in these meetings has been growing rapidly as is partly evidenced by the fact that the published reports contain 306, 450, and 755 pages, respectively.

The oldest mathematical society extant is the one at Hamburg, Germany, having been founded in 1690, and hence it has continued in activity over two hundred years. It can scarcely be called a national organization, especially since the founding of the Deutsche Mathematiker-Vereinigung in 1890, which holds its annual meetings in various parts of the German Empire in connection with the Gesellschaft Deutscher Naturforcher und Arzte. This has become the largest and one of the most influential mathematical societies of the world, even if its fees are only nominal,—2 marks per annum, or 30 marks for life membership. A considerable portion of its 700 members is composed of foreigners so that it, like the other leading societies, is partly international. It publishes a journal, which now appears monthly, and is perhaps best known on account of the extensive reports on the present state of various branches of higher mathematics, which have appeared in it from time to time. Members may receive this journal and occasional other publications at a reduced price.

The London Mathematical Society is the oldest among the prominent